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A SCRUTINY ON THE CUBIC EQUATION WITH FOUR UNKNOWNS

$$x^2 - y^2 = z^3 - w^3$$

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ABSTRACT

This paper aims at determining non-zero distinct integer solutions to the non-homogeneous cubic equation with four unknowns $x^2 - y^2 = z^3 - w^3$ by reducing it to the negative pellian equation through suitable transformations. A few relations among the solutions are presented.

KEYWORDS: Non-homogeneous cubic equation, Quaternary cubic equation, Integer solutions, Transformation technique.

INTRODUCTION

The theory of Diophantine equations offers a rich variety of fascinating problems. In particular, cubic diophantine equations, homogeneous and non-homogeneous have aroused the interest of numerous mathematicians since antiquity . In this context, one may refer [1-21] for various problems on the cubic diophantine equations .However, often we come across homogeneous and non-homogeneous cubic equations and as such one may require its integral solution in its most general form. It is towards this end, this paper concerns with the problem of determining a general form of non-trivial integral solutions of the non-homogeneous cubic equation with four unknowns given by $x^2 - y^2 = z^3 - w^3$.

A few relations among the solutions are presented.

Method of analysis

The non-homogeneous quaternary cubic equation to be solved for integer solutions is

$$x^2 - y^2 = z^3 - w^3 \tag{1}$$

Two different patterns of integer solutions to (1) are illustrated below:

Pattern 1

The option

$$\begin{aligned} x &= (2s^2 + 3s + 1)Y, \; y = (2s^2 + s)Y, \\ z &= (2s+1)(W+1), \; w = (2s+1)W \end{aligned} \tag{2}$$

in (1) leads to the negative pellian equation

$$\alpha^2 = 12Y^2 - 3 \tag{3}$$

$$\text{Where } \alpha = 6W + 3 \tag{4}$$

The smallest positive integer solution (Y_0, α_0) to (3) is given by

$$Y_0 = 1, \alpha_0 = 3$$

To obtain the other solutions to (3) , consider the pellian equation

$$\alpha^2 = 12Y^2 + 1 \tag{5}$$

whose least positive integer solution $(\tilde{Y}_0, \tilde{\alpha}_0)$ is given by

$$\tilde{Y}_0 = 2, \tilde{\alpha}_0 = 7$$

The general solution $(\tilde{Y}_n, \tilde{\alpha}_n)$ to (5) is given by

$$\tilde{\alpha}_n = \frac{1}{2} f_n, \tilde{Y}_n = \frac{1}{2\sqrt{12}} g_n$$

where

$$f_n = (7 + 2\sqrt{12})^{n+1} + (7 - 2\sqrt{12})^{n+1}, \; g_n = (7 + 2\sqrt{12})^{n+1} - (7 - 2\sqrt{12})^{n+1}$$

Employing the lemma of Brahmagupta between the solutions (Y_0, α_0) &$(\tilde{Y}_n, \tilde{\alpha}_n)$,

we have

$$Y_{n+1} = \frac{1}{2} f_n + \frac{3}{2\sqrt{12}} g_n, \; \alpha_{n+1} = \frac{3}{2} f_n + \frac{6}{\sqrt{12}} g_n \tag{6}$$

From (4), we have

$$W_{n+1} = \frac{1}{6}(\alpha_{n+1} - 3) \qquad (7)$$

Using the values of Y_{n+1}, W_{n+1} from (6) ,(7) in (2),the corresponding integer solutions to (1) are given by

$$\begin{aligned} x_{n+1} &= (2s^2 + 3s + 1)\,(\frac{1}{2} f_n + \frac{3}{2\sqrt{12}} g_n), \\ y_{n+1} &= (2s^2 + s)(\frac{1}{2} f_n + \frac{3}{2\sqrt{12}} g_n), \\ z_{n+1} &= (2s+1)\frac{1}{6}(\frac{3}{2} f_n + \frac{6}{\sqrt{12}} g_n + 3), \\ w_{n+1} &= (2s+1)\frac{1}{6}(\frac{3}{2} f_n + \frac{6}{\sqrt{12}} g_n - 3), n = 0,1,2,... \end{aligned} \qquad (8)$$

A few numerical solutions to (1) are presented in Table-1 below:

Table-1- Numerical solutions.

s	n	x_{n+1}	y_{n+1}	z_{n+1}	w_{n+1}
1	-1	6	3	3	0
	0	78	39	24	21
	1	1086	543	315	312
	2	15126	7563	4368	4365
2	-1	15	10	5	0
	0	195	130	40	35
	1	2715	1810	525	520
	2	37815	25210	7280	7275

Observations:

1. $x_{n+3} - 14x_{n+2} + x_{n+1} = 0$
2. $y_{n+3} - 14y_{n+2} + y_{n+1} = 0$
3. $z_{n+3} - 14z_{n+2} + z_{n+1} = -6(2s+1)$
4. $w_{n+3} - 14w_{n+2} + w_{n+1} = 6(2s+1)$
5. $15\,x_{2n+2} - x_{2n+3} + 4s^2 + 6s + 2 = (2s^2 + 3s + 1)$ times a perfect square
6. $15\,y_{2n+2} - y_{2n+3} + 4s^2 + 2s = (2s^2 + s)$ times a perfect square
7. $4(x_{n+2} - 13x_{n+1})^2 + 12(2s^2 + 3s + 1)^2 = 3\,(2s^2 + 3s + 1)^2$ times a perfect square
8. $3\,(15x_{n+1} - x_{n+2})^2 - 4(x_{n+2} - 13x_{n+1})^2 = 12\,(2s^2 + 3s + 1)^2$
9. $3(2s^2 + 3s + 1)(15x_{2n+2} - x_{2n+3}) - 4(x_{n+2} - 13x_{n+1})^2 = 6(2s^2 + 3s + 1)^2$

10. $2(z_{2n+3}-13z_{2n+2})+14(2s+1)=(2s+1)$ times a perfect square

11. $8(2s+1)\,x_{2n+2}-12(2s^2+3s+1)z_{2n+2}+8(2s+1)\,(2s^2+3s+1)=(2s+1)\,(2s^2+3s+1)$ times a perfect square

Pattern 2

The substitution

$$x=u+1,y=u-1,z=v+1,w=v-1,u,v\neq\pm1 \qquad (9)$$

in (1) gives

$$4u=6v^2+2$$
$$\Rightarrow u=\frac{(3v^2+1)}{2} \qquad (10)$$

Thus, taking

$$v=2s+1$$

in (10) , one has

$$u=6s^2+6s+2$$

In view of (9) ,one has the integer solutions to (1) to be

$$x=6\,s^2+6s+3\,,y=6\,s^2+6s+1$$
$$z=2s+2,w=2s \qquad (11)$$

A few numerical solutions are given in Table-2 below:

Table-2-Numerical solutions.

s	x	y	z	w
1	15	13	4	2
2	39	37	6	4
3	75	73	8	6
4	123	121	10	8

Observations

1. Each of the following expressions is a square multiple of six

(i) $x-3z+3$

(ii) $y-3z+5$

2. For values of s given by

$$s=s_n=\frac{[(5+2\sqrt{6})^{n+1}-(5-2\sqrt{6})^{n+1}]}{2\sqrt{6}} \quad ,$$

it is seen that $y-3w$ is a perfect square

3. For values of s given by

$$s=s_{n+1}=\frac{1}{2\sqrt{6}}\{\sqrt{6}\,[(5+2\sqrt{6})^{n+1}+(5-2\sqrt{6})^{n+1}]+3[(5+2\sqrt{6})^{n+1}-(5-2\sqrt{6})^{n+1}]\},$$

it is seen that $x-3w$ is a perfect square.

4. $yz=xw+2zw+w^2+2$

5. $2yz=3w^3+12w^2+14w+4$

CONCLUSION

An attempt has been made to obtain non-zero integer solutions to the non- homogeneous quaternary cubic Diophantine equation $x^2-y^2=z^3-w^3$ by reducing it to the solvable pellian equation through suitable transformations and the integer solutions are different from those of [21].One may search for other choices of transformations to reduce the degree of given equation to a lower degree equation that is solvable.

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