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ON CUBIC EQUATION WITH FOUR UNKNOWNS

$$x^3 + y^3 + 2(x+y)(x-y)^2 = 22zw^2$$

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ABSTRACT

The homogeneous cubic equation with four unknowns represented by the Diophantine equation $x^3 + y^3 + 2(x+y)(x-y)^2 = 22zw^2$ is analyzed for its patterns of non-zero distinct integral solutions.

KEYWORDS: Homogeneous cubic equation., Quaternary cubic, Integer solutions.

INTRODUCTION

The cubic diophantine equations are rich in variety and offer an unlimited field for research . For an extensive review of various problems, one may refer [1-21]. This paper concerns with another interesting cubic diophantine equation with four unknowns $x^3 + y^3 + 2(x+y)(x-y)^2 = 22zw^2$ for determining its infinitely many non-zero integral solutions.

METHOD OF ANALYSIS

The homogeneous cubic equation with four unknowns to be solved for its distinct non-zero integral solution is

$$x^3 + y^3 + 2(x+y)(x-y)^2 = 22zw^2 \quad (1)$$

Introduction of the linear transformations

$$x = u + v, y = u - v, z = u \qquad (2)$$

in (1) leads to

$$u^2 + 11v^2 = 11w^2 \qquad (3)$$

Different methods of obtaining the patterns of integer solutions to (1) are illustrated below:

PATTERN: 1

Equation (3) is written in the form of ratio as

$$\frac{u}{11(w+v)} = \frac{(w-v)}{u} = \frac{P}{Q}, Q \neq 0 \qquad (4)$$

which is equivalent to the system of double equations

$$Qu - 11Pv - 11wP = 0$$
$$Pu + Qv - wQ = 0$$

Applying the method of cross-multiplication to the above system of equations, we have

$$u = 22PQ, v = Q^2 - 11P^2, w = Q^2 + 11P^2 \qquad (5)$$

In view of (2) , one has

$$x = x(P,Q) = Q^2 - 11P^2 + 22PQ$$
$$y = y(P,Q) = 11P^2 - Q^2 + 22PQ$$
$$z = z(P,Q) = 22PQ$$

which satisfy (1) along with the value of w in (5).

PATTERN: 2

Let

$$w = a^2 + 11b^2 \qquad (6)$$

where a and b are non-zero integers.

Write 11 as

$$11 = \left(i\sqrt{11}\right)\left(-i\sqrt{11}\right) \qquad (7)$$

Using (5), (6) in (3) and applying the method of factorization, define

$$\left(u+i\sqrt{11}v\right)=\left(i\sqrt{11}\right)\left(a+i\sqrt{11}\,b\right)^2 \qquad (8)$$

from which we have

$$\left.\begin{aligned} u &= -22ab \\ v &= a^2-11b^2 \end{aligned}\right\} \qquad (9)$$

Using (9) and (2), the values of x, y and z are given by

$$\left.\begin{aligned} x &= x(a,b) = a^2-22ab-11b^2 \\ y &= y(a,b) = -a^2+11b^2-22ab \\ z &= z(a,b) = -14ab \end{aligned}\right\} \qquad (10)$$

Thus (6) and (10) represent the non-zero integer solutions to (1).

PATTERN: 3

Rewrite (3) as

$$u^2+11v^2=11w^2*1 \qquad (11)$$

Write 1 as

$$1=\frac{\left(5+i\sqrt{11}\right)\left(5-i\sqrt{11}\right)}{36} \qquad (12)$$

Assume

$$w=36(a^2+11b^2) \qquad (13)$$

Using (7), (12) ,(13) in (11) and applying the method of factorization, define

$$\left(u+i\sqrt{11}\,v\right)=\left(i\sqrt{11}\right)\left(\frac{5+i\sqrt{11}}{6}\right)\left(6a+i\,6\sqrt{7}b\right)^2$$

from which we have

$$\left.\begin{aligned} u &= 6\left(-11a^2+121b^2-110ab\right) \\ v &= 6\left(5a^2-55b^2-22ab\right) \end{aligned}\right\}$$

In view of (2), the corresponding integer solutions to (1) are given by

$$x = x(a, b) = 6[-6a^2 + 66b^2 - 132ab]$$
$$y = y(a, b) = 6[-16a^2 + 176b^2 - 88ab]$$
$$z = z(a, b) = 6[-11a^2 + 121b^2 - 110ab]$$
$$w = w(a, b) = 36\left(a^2 + 11b^2\right)$$

Note: 1

It is to be noted that, in addition to (12), 1 can be expressed as below:

$$1 = \frac{\left(11r^2 - s^2 + i\sqrt{11(2rs)}\right)\left(11r^2 - s^2 - i\sqrt{11}(2rs)\right)}{\left(11r^2 + s^2\right)^2}$$

$$1 = \frac{(2s^2 - 2s - 5 + i\sqrt{11}(2s-1))\ (2s^2 - 2s - 5 - i\sqrt{11}(2s-1))}{(2s^2 - 2s + 6)^2}$$

$$1 = \frac{(22s^2 - 22s + 5 + i\sqrt{11}(2s-1))\ (22s^2 - 22s + 5 - i\sqrt{11}(2s-1))}{(22s^2 - 22s + 6)^2}$$

Following the above procedure , three more sets of integer solutions to (1) are obtained.

PATTERN: 4

(3) can also be written as

$$11w^2 - u^2 = 11v^2 \qquad (14)$$

Let

$$11 = \frac{\left(6\sqrt{11} + 11\right)\left(6\sqrt{11} - 11\right)}{25} \qquad (15)$$

Take

$$v = 25(11a^2 - b^2) \qquad (16)$$

Using (15), (16) in (14) and employing the method of factorization and equating the positive parts separately, we get,

$$\left(\sqrt{11}\,w + u\right) = \left(5\sqrt{11}\,a + 5b\right)^2 \left(\frac{6\sqrt{11} + 11}{5}\right)$$

From which we have

$$\begin{aligned} w &= 5[6\,(11a^2 + b^2) + 22ab] \\ u &= 5[11\,(11a^2 + b^2) + 132ab] \end{aligned} \qquad (17)$$

In view of (2), we have the integer solutions to (1) given by

$$x = 880\,a^2 + 30\,b^2 + 660\,ab$$
$$y = 330\,a^2 + 80\,b^2 + 660\,ab$$
$$z = 605\,a^2 + 55\,b^2 + 660\,ab$$

along with the value of w given in (17).

PATTERN: 5

Taking

$$u = 11U \quad (18)$$ in (3) , one has

$$w^2 = 11U^2 + v^2 \quad (19)$$

Expressing (19) as the system of double equations

$$w + v = 11U^2$$
$$w - v = 1$$

And solving, we get

$$U = 2s + 1$$
$$v = 22s^2 + 22s + 5 \quad (20)$$
$$w = 22s^2 + 22s + 6$$

From (18), we get

$$u = 22s + 11$$

Substituting the above values of u and v in (2), we have

$$x = 22s^2 + 44s + 16$$
$$y = -22s^2 + 6 \quad (21)$$
$$z = 22s + 11$$

Thus, (1) is satisfied by (21) alongwith the value of w in (20).

Note: 2

Expressing (19) as the system of double equations

$$w + v = U^2$$
$$w - v = 11$$

and proceeding as above , the corresponding integer solutions to (1) are given by

$$x = 2s^2 + 24s + 6$$
$$y = -2s^2 + 20s + 16$$
$$z = 22s + 11$$
$$w = 2s^2 + 2s + 6$$

CONCLUSION

In this paper, we have made an attempt to determine different patterns of non-zero distinct integer solutions to the homogeneous cubic equation with four unknowns. As the cubic equations are rich in variety, one may search for other forms of cubic equations with multi-variables to obtain their corresponding solutions .

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