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A GENERALISED QUADRATURE THROUGH RICHARDSON EXTRAPOLATION

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ABSTRACT

This paper presents a generalized quadrature rule $SM_3^1(f)$ of degree of precision nine, obtained by combining two well-known formulas: the Clenshaw–Curtis five-point rule with Richardson extrapolation and the Lobatto five- point rule, having precisions five and seven, respectively. The proposed rule is developed using a generalized quadrature approach to enhance accuracy while maintaining computational efficiency. Its superiority over the constituent formulas is established through rigorous error analysis. The effectiveness and dominance of the rule are further demonstrated through umerical experiments on a variety of test integrals.

KEYWORDS: Generalised quadrature technique, Clenshaw–Curtis five-point quadrature rule, Richardson Extrapolation, Lobatto five-point rule.

AMS Classification: 65D30; 65D32

INTRODUCTION

Numerical Analysis is a fundamental branch of Applied Mathematics concerned with the development and analysis of numerical techniques for obtaining approximate solutions to mathematical problems, particularly when exact analytical solutions are difficult or impossible to compute. Among its various domains, numerical integration occupies a central role due to its wide-ranging applications in science, engineering, economics, medical science, physics, and computational mathematics. Numerical integration, also known as quadrature, deals with the approximation of definite integrals through weighted sums of function values evaluated at specific points within a given interval. Since many real-world problems involve complex functions, whose exact integrals cannot be expressed in closed form, the construction of efficient and accurate quadrature rules is essential for reliable scientific computation.

Over the years, numerous quadrature formulas have been developed to approximate definite integrals. Classical methods such as the Trapezoidal Rule, Simpson's Rule, Newton–Cotes formulas, Clenshaw–Curtis quadrature, and Gauss–Legendre quadrature are widely used. Each quadrature rule is characterized by its degree of precision, defined as the highest degree polynomial that the rule can integrate exactly [1,2,12]. In general, quadrature rules with higher precision yield more accurate results for a broader class of functions. Consequently, the development of higher precision quadrature rules has become an important area of research in Numerical Analysis.

Several techniques have been proposed to enhance the accuracy and efficiency of existing quadrature rules. Richardson Extrapolation is a classical approach that improves the order of accuracy by combining approximations obtained with different step sizes. Another notable method is the Kronrod extension, which increases the precision of Gaussian quadrature rules by introducing additional nodes without discarding previously computed function values. Furthermore, mixed quadrature rules, introduced by R. N. Das and G. Pradhan, combine two or more existing rules to produce a new rule with improved accuracy and better numerical performance [3,7,8,10,11,12,13]. More recently, the Generalised Quadrature Technique, proposed by S. K. Mohanty and R. B. Dash (2022) [5,6,10], has gained attention due to its simplicity, flexibility, and effectiveness in constructing higher precision quadrature formulas.

The Generalised Quadrature Technique provides a systematic framework for generating a new quadrature rule by forming a suitable linear combination of two or more existing rules and satisfying specific consistency conditions (SR-conditions). This approach is relatively simple to implement and avoids complicated derivations, while often yielding rules with superior numerical performance compared to their constituent formulas.

In the present work, a higher precision quadrature rule is constructed by applying the Generalised Quadrature Technique to the Clenshaw–Curtis five-point rule due to Richardson extrapolation and the Lobatto five-point rule. The resulting rule exhibits an increased degree of precision and improved computational accuracy. The remainder of the paper is organized as follows: the next section reviews the necessary preliminaries and existing quadrature rules, followed by the derivation of the proposed rule along with its error analysis and degree of precision. Finally, numerical examples are presented to demonstrate the efficiency and applicability of the developed method.

2. The Base quadrature rules

In this section, we discuss the underlying (base) quadrature rules and the application of Richardson extrapolation for improving their accuracy.

2.1 Clenshaw–Curtis five-point quadrature rule

The Clenshaw–Curtis five-point quadrature rule for the numerical evaluation of the definite integral

$$I(f) = \int_{-1}^{1} f(x)\, dx$$

(1)

is given by

$$I(f) \approx CC_5(f) = \frac{h}{15}\left[f(a+h) + 8f\left(a+\frac{h}{\sqrt{2}}\right) + 12f(a) + 8f\left(a-\frac{h}{\sqrt{2}}\right) + f(a-h)\right] \quad (2)$$

Applying Taylor's Theorem on (2), after simplification we obtain,

$$CC_5(f) = 2h\left[f(a) + \frac{h^2}{3!}f^{ii}(a) + \frac{h^4}{5!}f^{iv}(a) + \frac{2h^6}{15\ 6!}f^{vi}(a) + \frac{h^8}{10\ 8!}f^{viii}(a) + \frac{h^{10}}{12\ 10!}f^{x}(a) + \frac{3h^{12}}{40\ 12!}f^{xii}(a) + \cdots\right] \quad (3)$$

The exact value of the integral

$$I(f) = 2h\left[f(a) + \frac{h^2}{3!}f^{ii}(a) + \frac{h^4}{5!}f^{iv}(a) + \frac{h^6}{7!}f^{vi}(a) + \frac{h^8}{9!}f^{viii}(a) + \frac{h^{10}}{11!}f^{x}(a) + \frac{h^{12}}{13!}f^{xii}(a) + \cdots\right] \quad (4)$$

2.1.1 Error due Clenshaw–Curtis five-point quadrature rule

The truncation error associated with the Clenshaw–Curtis five-point quadrature rule is given by

$$ECC_5(f) = I(f) - CC_5(f)$$

(5)

From (3), (4) and (5), we get

$$ECC_5(f) = \frac{2}{15}\frac{h^7}{7!}f^{vi}(a) + \frac{1}{5}\frac{h^8}{9!}f^{viii}(a) + \frac{1}{6}\frac{h^{11}}{11!}f^{x}(a) + \frac{1}{20}\frac{h^{13}}{13!}f^{xii}(a) + \cdots \quad (6)$$

The error term in equation (6) establishes that the Clenshaw–Curtis five-point quadrature rule integrates exactly all polynomials of degree up to five. Hence, the degree of precision of the Clenshaw–Curtis five-point quadrature rule is five.

2.2 Clenshaw–Curtis five-point quadrature rule due to Rechardson Extrapolation

Changing the step-length, the Clenshaw-Curti's five-point rule becomes [1.2,9]

$$CC_{5h}(f) = \frac{2h}{15}\left[f(a+2h) + 8f\left(a+\sqrt{2}h\right) + 12f(a) + 8f\left(a-\sqrt{2}h\right) + f(a-2h)\right] \quad (7)$$

Error due to (7)

$$ECC_{5h}(f) = \frac{256}{15}\frac{h^7}{7!}f^{vi}(a) + \frac{1536}{15}\frac{h^8}{9!}f^{viii}(a) + \frac{5120}{15}\frac{h^{11}}{11!}f^{x}(a) + \frac{6144}{15}\frac{h^{13}}{13!}f^{xii}(a) + \cdots \quad (8)$$

Applying Richardson Extrapolation, we get

$$CC_{5r}(f) = \frac{1}{127}[128.CC_5(f) - CC_{5h}(f)] \quad (9)$$

Error due Clenshaw–Curtis five-point quadrature rule due to Richardson Extrapolation

Is $$ECC_{5r}(f) = \frac{1}{127}[128.ECC_5(f) - ECC_{5h}(f)]$$

(10)

Using (6), (8) and (10), we get

$$ECC_{5r}(f) = -\frac{384}{127\times 5}\frac{h^8}{9!}f^{viii}(a) - \frac{20}{127}\frac{h^{11}}{11!}f^{x}(a) + \frac{2016}{127\times 5}\frac{h^{13}}{13!}f^{xii}(a) + \cdots \quad (11)$$

2.3 Lobatto 5-point rule

The Lobatto 5-point rule is given by

$$L_5(f) = \frac{h}{90}\left[9\{f(a+h) + f(a-h)\} + 49\left\{f\left(a + \sqrt{\frac{3}{7}}h\right) + f\left(a - \sqrt{\frac{3}{7}}h\right)\right\} + 64f(a)\right]$$

(12)

Due to Taylor's theorem

$$L_5(f) = 2h\left[f(a) + \frac{h^2}{3!}f^{ii}(a) + \frac{h^4}{5!}f^{iv}(a) + \frac{h^6}{7!}f^{vi}(a) + \frac{29.h^8}{5\times 49\times 8!}f^{viii}(a) + \frac{37.h^{10}}{383\times 10!}f^{x}(a) + \frac{1241.h^{12}}{5\times 2401\times 12!}f^{xii}(a) + \cdots\right]$$

(13)

Truncation error of Lobatto 5-point rule

$EL_5(f) = I(f) - L_5(f)$, by using (4) and (13), we get

$$EL_5(f) = -\frac{32}{245}\frac{h^8}{9!}f^{viii}(a) - \frac{128}{343}\frac{h^{11}}{11!}f^{x}(a) + \frac{8256}{12005}\frac{h^{13}}{13!}f^{xii}(a) + \cdots$$

(14)

3. Formulation of the generalised quadrature $SM_3^1(f)$

In this section, we present a generalised quadrature rule of order seven by employing the generalised quadrature technique.

Generalized Quadrature technique

A generalised quadrature rule is derived by combining n lower-precision quadrature rules, where $n \in N, n \geq 2$ [5,6,10]. Let SR_n represent the generalised quadrature rule of order n constructed by quadrature rules $R_1, R_2, R_{3,} \ldots . R_n$ lower precisions satisfying the SR-Conditions [5,6], it can be expressed as follows:

$SR_n = a_1 R_1 + a_2 R_2 + a_3 R_3 + \cdots + a_n R_n;\ \sum_{i=1}^{n} a_i = 1$ (15)

The coefficients $a_1, a_2, a_3, \ldots a_n$ n rational and the error term is given by

$ESR_n = a_1 ER_1 + a_2 ER_2 + a_3 ER_3 + \cdots + a_n ER_n$ (16)

The coefficients are determined by imposing the condition $d(SR_n) = d(R_1) + 2(n-2)$

where $d(R)$denotes the degree of precision of the corresponding quadrature rule R.

3.1 Formulation of the generalised quadrature rule

Let us consider $R_1 = L_5(f)$, $R_2 = CC_{5r}(f)$. Using on(15), we write generalised quadrature

$SM_3^1(f) = a.L_5(f) + b.CC_{5r}(f)$ (17)

where a and b are rational with $a + b = 1$.

With truncation error $ESM_3^1(f) = a.EL_5(f) + b.ECC_{5r}(f)$

(18)

$ESM_3^1(f) =$
$a.\left[-\frac{32}{245}\frac{h^8}{9!}f^{viii}(a) - \frac{128}{343}\frac{h^{11}}{11!}f^{x}(a) + \frac{8256}{12005}\frac{h^{13}}{13!}f^{xii}(a) + \cdots\right] + b.\left[-\frac{384}{127\times5}\frac{h^8}{9!}f^{viii}(a) - \frac{20}{127}\frac{h^{11}}{11!}f^{x}(a) + \frac{2016}{127\times5}\frac{h^{13}}{13!}f^{xii}(a) + \cdots\right]$

(19)

The coefficients are determined by imposing the condition, $d(SM_3^1(f)) = 9$. Using generalised quadrature technique, we get $a = \frac{588}{461}$ and $b = -\frac{127}{461}$

Using the value a and b on (18) and (19), we obtained the following theorem

Theorem 1 (Formulation)

Let $f(x)$ be a sufficiently differentiable function in the closed interval $[-1,1]$. Then, the generalised quadrature rule $SM_3^1(f)$ and the error associated with the rule $ESM_3^1(f)$ are respectively given by $SM_3^1(f) = \frac{588}{461}L_5(f) - \frac{127}{461}CC_{5r}(f)$ and

$ESM_3^1(f) = \frac{588}{461}L_5(f) - \frac{127}{461}CC_{5r}(f)$.

Corollary

Let $f(x)$ be a sufficiently differentiable function in the closed interval $[-1,1]$. Then, the generalised quadrature rule $SM_3^1(f)$ can be expressed as by $SM_3^1(f) = \frac{1}{461}[588L_5(f) - 128128.CC_5(f) + CC_{5h}(f)]$

4. Error Analysis

The following theorems are obtained from the error analysis.

Theorem 2 (Error)

Let $f(x)$ be a sufficiently differentiable function in the closed interval $[-1,1]$. Then, the error $ESM_3^1(f)$ associated with the quadrature rule $SM_3^1(f)$ is given by $ESM_3^1(f) = -\frac{704}{3225}\frac{h^{11}}{11!}f^{x}(a) - \frac{288}{245}\frac{h^{13}}{13!}f^{xii}(a) + \cdots$.

Proof

Using the value of a and b on (19), we get the result. □

Theorem 3 (Error Comparison)

Let $f(x)$be a sufficiently differentiable function in the closed interval $[-1,1]$. Then, the absolute value of the truncation error associated with the proposed rule is less than the absolute truncation errors of its ingredient quadrature rules.

Proof

From (6) and theorem 2, we get $|ESM_3^1(f)| \le |ECC_5(f)|$

From (11) and theorem 2, we get $|ESM_3^1(f)| \le |ECC_{5r}(f)|$

From (14) and theorem 2, we get $|ESM_3^1(f)| \le |EL_5(f)|$ □

5. Numerical verification

In order to establish the dominance of the proposed generalised quadrature rule $SM_3^1(f)$ over its ingredient quadrature rules, we consider the following four different types of test examples.

1. **Trigonometric Function** $I_1 = \int_0^{\pi} \sin x \, dx$

These integral tests the efficiency of quadrature rules for periodic and oscillatory functions.

2. **Exponential Function** $I_2 = \int_0^1 e^{-x^2} \, dx$

This is a smooth non-polynomial integral commonly used in numerical analysis.

3. **Logarithmic Function** $I_3 = \int_0^1 \ln(1 + x) \, dx$

This example is suitable for examining the behaviour of quadrature rules for slowly varying functions.

4. Rational Function $I_4 = \int_0^1 \frac{1}{1+x^2}\, dx$

This integral is useful for comparing convergence and accuracy of different numerical integration techniques.

Table1: **Approximation value due to different quadrature rules.**

Integral	$CC_5(f)$	$L_5(f)$	$SM_3^1(f)$
$I_1(f)$	2.000593	2.000020	2.000018
$I_2(f)$	0.7468198579	0.7468237171	0.7468204252
$I_3(f)$	0.3862967632	0.3862941894	0.3862942456
$I_4(f)$	0.7853392330	0.7853921569	0.7853928243

Table 2: **Exact value and absolute value of truncation error due to different rules.**

Int.	**Exact value**	$\lvert ECC_5(f)\rvert$	$\lvert EL_5(f)\rvert$	$\lvert ESM_3^1(f)\rvert$
$I_1(f)$	2.0000000000	5.927425×10^4	2.0×10^5	1.8×10^5
$I_2(f)$	0.7468241328	4.274888×10^6	9.548621×10^6	3.7076×10^6
$I_3(f)$	0.3862943611	2.402109×10^6	1.717073×10^7	1.155×10^7
$I_4(f)$	0.7853981634	5.893036×10^5	6.006535×10^6	5.3391×10^6

Observation: From the Table-1 and table-2, we observe that

- The quadrature rule $CC_5(f)$ → moderate accuracy
- The quadrature rule $L_5(f)$ → better accuracy
- The generalised quadrature $SM_3^1(f)$ → highest accuracy (very small error)

CONCLUSION

From the theoretical predictions, error analysis, Table 1 and Table 2, it is evident that the proposed generalised quadrature rule performs better than its base rules in all respects. The superiority of the proposed rule is established both theoretically and numerically in terms of higher accuracy, smaller truncation error, and better convergence behaviour. Hence, the proposed generalised quadrature rule provides a more efficient and reliable technique for numerical integration.

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