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APPLICATION OF FRACTIONAL CALCULUS BASED ON MATHEMATICAL ECONOMICS

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Graphical ABSTRACT

The graphical abstract visually contrasts the response of an economic system modelled with classical integer-order calculus versus one modelled with fractional calculus.

Panel 1: Integer-Order Model: "Amnesiac" Response

A diagram representing a simple economic system at equilibrium. A vertical arrow signifies an external shock or policy intervention (e.g., a fiscal stimulus). The system's immediate response is depicted as a sharp, instantaneous change followed by a rapid, exponential decay back to its original or a new equilibrium state. This model's response is purely local and dependent on the state now of the shock.

Panel 2: Fractional-Order Model: Response with "Fading Memory"

The same initial diagram with the identical external shock. However, the system's response is a more gradual, persistent, and "fading" trajectory. The path is influenced by the entire history of the system's evolution, not just the instantaneous shock. This trajectory is characterized by a power-law decay, which demonstrates a longer-term dependence on past states. The system eventually converges to an equilibrium, but its path is non-local, and its long-term dynamics are inherently different from the classical model.

This visual comparison encapsulates the central thesis of the paper: fractional calculus provides a framework to move beyond the assumption of economic "amnesia" and model systems with long-term memory.

ABSTRACT

This paper provides an exhaustive analysis of the application of fractional calculus in mathematical economics, arguing for its necessity in modelling real-world economic phenomena characterized by long-term memory and non-local interactions. Classical economic models, largely built upon integer-order differential equations, suffer from a fundamental limitation often termed "amnesia," as they assume instantaneous, local dependencies and disregard the influence of historical states on current behaviour. Drawing from a comprehensive literature review, the analysis establishes fractional calculus as the mathematical framework for a new "Memory Revolution" in economics. A rigorous conceptual foundation is provided, differentiating between key fractional operators like the Riemann-Liouville and Caputo derivatives and offering a compelling economic interpretation of the fractional order as a measure of fading memory. To demonstrate its utility, a fractional-order extension of the canonical Solow-Swan growth model is formulated. Through analytical and numerical methods, the analysis shows that the fractional model significantly affects the trajectory and long-term stability of capital accumulation, providing a more flexible and realistic representation of economic dynamics. The empirical evidence, including superior data-fitting performance in GDP modelling, validates the approach. The paper concludes by addressing the challenges of this nascent field and outlining future research directions, positioning fractional mathematical economics as a vital and emerging discipline.

KEYWORDS: Fractional Calculus, Mathematical Economics, Non-Locality, Long Memory, Caputo Derivative, Riemann-Liouville Derivative, Economic Growth Models, Solow-Swan Model, Option Pricing, Fading Memory, Anomalous Diffusion.

1. INTRODUCTION

1.1. Contextual Background: The Rise of Calculus in Economic Theory

The formalization of modern economics has been inextricably linked to the adoption of mathematical tools. The late 19th century's "Marginal revolution" and the early 20th century's "Keynesian revolution" introduced fundamental concepts like "marginal value," "economic multiplier," "economic accelerator," and "elasticity" into economic theory [1]. These concepts were, by their nature, mathematically expressed using the tools of integer-order differential and integral calculus, which provided a powerful and elegant framework for describing economic phenomena in both continuous and discrete time [2]. The application of these mathematical methods allowed for the development of models that could analyse and predict the behaviour of key economic indicators, laying the groundwork for the field of mathematical economics as it is known today.

1.2. The Problem with "Amnesia": A Critique of Classical Assumptions

Despite their widespread success and acceptance, these classical models operate under a crucial and often-unspoken assumption: that economic systems have a form of "complete amnesia" [3-4]. This is a direct consequence of the properties of integer-order derivatives, which are determined by the behaviour of functions in an infinitely small neighbourhood of a point. As a result, differential equations with integer-order derivatives cannot describe processes with memory or non-locality. They model economic interactions as if all agents have no memory of past events and only interact with their immediate surroundings. This assumption of instantaneous and local dependence profoundly contradicts real-world economic behaviour.

The limitations of models built on these assumptions have been highlighted by various schools of economic thought. Institutional economics, for example, criticized the classical theory of value for its neglect of concepts like scarcity, ownership, and the role of institutions and social norms. Behavioural economics has further demonstrated that individuals are not perfectly rational actors but often make decisions based on heuristics, biases, and limited information, which can lead to market failures [5-8]. However, the fragmented nature of these critiques, which span from the institutional to the behavioural and now the mathematical realm, can be unified under a common intellectual core: the inadequacy of models that assume "amnesia." The inability to account for the way past events influence current and future states is not merely a technical oversight; it represents a profound conceptual flaw in the foundational assumptions of classical economics. This is the central problem that fractional calculus is uniquely positioned to address.

1.3. A New Paradigm: The Fractional Calculus "Memory Revolution"

Fractional calculus (FC), a branch of mathematics dealing with derivatives and integrals of non-integer order, provides the main mathematical tool to "cure amnesia" in economics [9-12]. Un like their integer-order counterparts, fractional operators are inherently non-local. This means that their value at a given point is influenced not just by the local conditions but by the entire history of the function's evolution [13-15]. This "hereditary property” allows for the direct incorporation of memory and long-range dependence into economic models, offering a more realistic representation of phenomena where past states influence current behavior. The emergence of this field, termed "fractional mathematical economics," constitutes a new stage in the evolution of economic theory, complementing the earlier revolutions by introducing the concepts of memory and non-locality.

1.4. Paper Objectives

This paper aims to provide a comprehensive, expert-level report on the application of fractional calculus in mathematical economics. The analysis will (i) review the historical and conceptual foundations of fractional calculus and its application in economics; (ii) establish a rigorous framework by defining key fractional operators and providing a nuanced economic interpretation of their parameters; (iii) demonstrate the practical utility of this framework through a detailed application to the canonical Solow-Swan growth model; and (iv) present and analyse the resulting insights, concluding with a discussion of the field's limitations and future directions.

2. Literature Review

2.1. Historical and Conceptual Foundations

The history of fractional calculus traces back to a 1695 correspondence between Leibniz and L'Hôpital, where the latter posed the question of a half-order derivative. For centuries, it remained a theoretical curiosity, a "theoretical branch of math" with a nascent body of work.[10] Its use in practical applications was sparse until the past few decades when many scientific fields, including physics, chemistry, engineering, and finance, began applying fractional differential equations to real-world problems. This long latency period, where a rigorous mathematical concept existed without a compelling application, parallels the historical lag between theoretical mathematical developments and their practical adoption in economics. Just as integer-order calculus took centuries to become central to economic thought, fractional calculus is now following a similar path. The delay was not due to a lack of mathematical rigor, but rather the absence of a clear, compelling *application* that demonstrated its necessity. The problem of economic "amnesia" provides that necessary application. Over time, several definitions of the fractional derivative have been proposed, including those by Grünwald-Letnikov, Riemann-Liouville, and Caputo.

2.2. Critiques of Classical Economics and the Case for Memory

Classical and neoclassical economic models have faced substantial criticism for their simplifying assumptions, such as perfect rationality, perfect information, and market efficiency, which rarely hold in the real world. Institutional economists, such as John R. Commons, criticized the classical theory of value for neglecting fundamental concepts like proprietary scarcity and the role of institutions in price determination. Similarly, behavioural economics has incorporated psychological insights to show that human behaviour often deviates from the rational calculations assumed by neoclassical models.

However, the most profound and unifying critique comes from the inability of these models to capture "memory" effects. Because of using integer-order derivatives, which are local operators, these models cannot account for the path-dependent nature of economic processes, where the current state is influenced by the entire history of the system. Economic processes such as investment decisions, consumption patterns, and technological adoption are not instantaneous; they exhibit long-term dependencies where past states influence present behavior. The failure to integrate these hereditary properties represents a fundamental disconnect between classical models and the complexity of real-world economic systems.

2.3. Emergence of Fractional Mathematical Economics

The application of fractional calculus in economics is part of a broader "Memory Revolution". The first stage of this revolution is associated with the work of Clive W. J. Granger, a Nobel laureate who introduced the concept of long memory and long-range dependence in time series data. This led to the development of statistical models like ARFIMA (Autoregressive Fractionally Integrated Moving Average).[1] Since then, the field has evolved through several stages, including the application of fractional Brownian motion, tectonophysics, and deterministic chaos.

Fractional mathematical economics is now emerging as an independent science, distinct from being merely a branch of applied mathematics. Its purpose is to use fractional calculus not only to solve existing economic problems but also to formulate "new economic concepts, notions, effects and phenomena". The development of fractional calculus itself is now being influenced by the specific goals and objectives arising from its application in economics and other sciences.

2.4. Applications in Macroeconomic Modelling

A significant body of research has applied fractional calculus to model macroeconomic aggregates, particularly Gross Domestic Product (GDP). Studies have shown that fractional-order models provide a more accurate and realistic representation of economic growth compared to their classical, integer-order counterparts. For instance, research on the Spanish economy and the G7 countries has empirically demonstrated that fractional models consistently exhibit superior performance.

This superior performance is not merely a theoretical claim; it is supported by quantitative evidence. A comparative analysis of integer-order and fractional-order models for GDP growth shows that the fractional approach yields better data-fitting results. This is measured by performance indices such as the Mean Squared Error (MSE), Mean Absolute Deviation (MAD), and the coefficient of determination (R^2). The fact that these models achieve higher R^2 values and lower errors suggests that the "memory" encoded by the fractional derivative is not a mere mathematical construct but a genuine, measurable property of real-world economic systems. The following table illustrates this quantitative advantage, providing crucial empirical validation for the theoretical argument.

Index / Models	**Integer Model (8)**	**Fractional Model (9)**
Statistic		
**MSE * 10^5 **	6.084	1.320
R^2	0.9920	0.9983
**MAD * 10^2 **	2.0820	0.9257

The table demonstrates that the fractional model significantly outperforms the integer model across all metrics, with a dramatic reduction in error and an increase in explanatory power. This empirical success provides a strong justification for the use of fractional calculus in macroeconomic modelling.

2.5. Applications in Financial Markets

The limitations of classical models are also acutely felt in financial markets. The foundational Black-Scholes model for option pricing, for example, relies on the assumption of Gaussian price fluctuations. This simplification fails to account for the "fat tails" and large, discontinuous price jumps, or "black swan" events, that are frequently observed in financial data. These events can lead to significant losses for hedging strategies based on the Black-Scholes model.

Fractional models provide a more robust and realistic framework for financial applications. Models based on fractional Brownian motion or Lévy processes can capture long-range autocorrelations and the possibility of large price jumps, offering a more reliable hedge against dramatic market drops. The fractional order parameter provides an additional degree of freedom, allowing for a more flexible method of describing asset behaviour and risk. For instance, a fractional Black-Scholes model can be used to adjust the premium payment mechanism for agricultural insurance based on a fractional order parameter, ensuring greater dynamism and flexibility. The field is also exploring concepts such as anomalous diffusion to estimate stock volatility, which provides a more nuanced understanding of market dynamics by differentiating between sub diffusion, super diffusion, and normal diffusion.

3. Basic Concept

3.1. Definitions of Key Fractional Operators

Fractional calculus is a generalization of integer-order calculus, extending the concepts of differentiation and integration to any real or complex order. The most used operators in economic modelling are the Riemann-Liouville and Caputo derivatives.

The **Riemann-Liouville (RL) fractional integral** of order n ε R is defined as:

$$K_a^n f(t) = \frac{1}{\Gamma(n)} \int_a^t (t-u)^{n-1} f(u) du \quad \text{for} \quad a \le t \le b$$

The **Riemann-Liouville (RL) fractional derivative** is then defined as an integer-order derivative of the fractional integral:

$$RLD_a^\alpha f(t) = \frac{d^m}{dt^m}\left(\frac{1}{\Gamma(m-\alpha)} \int_a^t (t-\tau)^{m-\alpha-1} f(\tau) d\tau\right)$$

The **Caputo fractional derivative** is defined differently, as a fractional integral of an integer-order derivative:

$$CD_a^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)} \int_a^t (t-\tau)^{m-\alpha-1} f^{(m)}(\tau) d$$

The choice between these two operators is a critical methodological decision driven by practical considerations. While the RL derivative was historically the first, the Caputo derivative is generally preferred for economic modelling and other pragmatic problems. This is because the Caputo derivative handles initial conditions in the classical, integer-order sense, which is consistent with the initial states of real-world systems. This property significantly facilitates both theoretical and numerical analyses of initial value problems, making it a more suitable tool for applied research in economics.

3.2. Economic Interpretation of the Fractional Order

The central innovation of fractional calculus is the concept of non-locality. Unlike classical derivatives, whose value depends only on local conditions in an infinitesimal neighbourhood of a point, fractional derivatives account for the entire history of the system's evolution.

The fractional order, α, provides an additional degree of freedom to fit a specific behaviour and, crucially, has a direct economic interpretation.

The fractional derivative can be seen as an interpolation between the standard average and marginal values of economic indicators. A proposed "T-indicator" allows for a generalization of these concepts, where the standard average and marginal values are special cases. When the order of the fractional derivative is zero ($\alpha=0$).

he T-indicator simplifies to the standard average value. When the order is one α, it simplifies to the standard marginal value. The values of α between zero and one allow for the consideration of a whole spectrum of intermediate values, providing a rich framework for describing economic processes with a "fading memory". This interpretation transforms the mathematical parameter into a meaningful economic concept, arguing that the models are not just fitting a curve but capturing a genuine, underlying economic property. This provides a powerful counterargument to the critique that fractional models are merely "nice fitting tools". By giving the fractional order a concrete economic meaning, the approach moves beyond a purely correlational analysis to a more explanatory one.

3.3. Key Mathematical Properties

A defining characteristic of fractional operators is the violation of many standard calculus properties. These include the Leibniz product rule, the chain rule, and the semi-group property. The violation of these rules, particularly the standard form of the Leibniz rule, is a characteristic property of derivatives of non-integer orders. While this non-standard behaviour complicates analysis, it is precisely what allows fractional derivatives to capture the complex, non-local dynamics of systems with memory. For example, the non-standard product rule for fractional derivatives is a known obstacle that requires specialized methods to overcome. To solve fractional differential equations, a key mathematical tool is the Mittag-Leffler function, which serves as a generalization of the exponential function and is crucial for obtaining analytical solutions.

4. Materials and Methods

4.1. Model Selection: The Solow-Swan Framework

The Solow-Swan model, or exogenous growth model, is a foundational model in modern economic growth theory that provides key insights into capital accumulation and long-run economic growth. It is a cornerstone of economic education and serves as a starting point for numerous theoretical and applied extensions. The model's classical form is governed by a single, nonlinear ordinary differential equation, which makes it an ideal candidate to demonstrate the limitations of integer-order derivatives and the power of their fractional extension.

4.2. Classical vs. Fractional Formulation

The classical Solow-Swan model is governed by the following first-order differential equation:

$$\frac{dk(t)}{dt} = k(t)(pk^{\mu-1}(t) - q), \quad k(0) = k_0$$

Here, k(t) = K(t)/L(t) represents the capital-to-labour ratio over time, with K as capital and L as labour. The parameters p and q are positive constants, where p relates to productivity and q to the depreciation rate. This traditional formulation assumes that the rate of change of capital at time t depends only on the system's state at that instant, implying a memoryless process.

To introduce memory effects, the analysis formulates a fractional version of this model by substituting the integer-order derivative with a Caputo fractional derivative of order $\alpha, where\ 0 < \alpha \leq 1$. This yields the fractional Solow-Swan model:

$$D_t^{\alpha} k(t) = k(t)(pk^{\mu-1}(t) - q), \quad k(0) = k_0$$

This formulation explicitly accounts for the influence of past states on the present rate of capital change. The Caputo derivative is specifically chosen for this purpose because its treatment of initial conditions in the classical sense is well-suited for economic applications and facilitates both theoretical and numerical analysis. The fractional model is a generalization of the classical model, as it reduces to the integer-order form when α= 1.

4.3. Analytical and Numerical Approach

Solving fractional differential equations is often more complex than solving their integer-order counterparts and closed-form solutions can be elusive. Therefore, a combination of analytical and numerical methods is employed.

For analytical solutions, integral transforms such as the Sumudu Transform are utilized. The Sumudu Transform is a linear integral transform with desirable properties, including unit preservation and domain scaling, which make it attractive for solving fractional differential equations.

For numerical simulations, algorithms like the Adams-Bashforth-Moulton algorithm are used to approximate the solutions. This numerical approach is crucial for visualizing the model's behaviour under different parameters and for scenarios where an exact analytical solution is not feasible. These need for advanced computational methods highlights a significant trade-off: while fractional calculus offers a more realistic and flexible modelling framework, this comes at the cost of increased computational complexity compared to the highly tractable classical models. Acknowledging this trade-off is essential for a balanced and comprehensive analysis of the approach.

5. Example, Result

5.1. Model Parameterization and Analysis

The analysis begins with a systematic examination of the fractional Solow-Swan model's behaviour under various fractional orders α and key scaling parameters (p and q). The fractional order itself dictates the strength of the memory effect. As α increases from values close to 0 towards 1, the system transitions from strong memory effects to a behaviour resembling the classical, memoryless Solow-Swan model. Smaller values of α lead to more gradual capital accumulation due to historical inertia, providing a more flexible and comprehensive framework for modelling economic growth.

The influence of the depreciation rate (q) is also explored. As the depreciation rate increases, the growth of capital per labour diminishes. In the fractional model, the memory effect introduced by α leads to smoother transitions and more gradual capital dynamics compared to the classical model, which exhibits a more rapid, exponential decay.[27] Similarly, the analysis shows that higher productivity values (p) significantly enhance capital accumulation, while lower values result in slower or stagnant growth.[27] The fractional model captures the persistent influences of historical investment decisions and shows smoother transitions in response to changes in productivity.

5.2. Comparative Analysis and Key Findings

The central finding of this analysis is that the fractional-order model provides a more realistic and nuanced representation of economic growth dynamics compared to the classical integer-order model. This is demonstrated through a direct comparison of their time histories and long-term behaviour. While both models may eventually converge to a similar equilibrium, the fractional model exhibits a more gradual, path-dependent trajectory, reflecting the influence of historical context.

The following figure provides a conceptual illustration of the capital-labour ratio's time history under both classical and fractional formulations.

Capital-Labor Ratio Dynamics in Classical vs. Fractional Solow-Swan Models shows that the classical model (represented by α) converges quickly to its steady state. In contrast, the fractional models, with $\alpha < 1$, exhibit a more prolonged and gradual convergence. The trajectory is influenced by the entire history of capital accumulation, as dictated by the fractional derivative's non-local nature. The inclusion of a fractional-order derivative significantly affects the trajectory and long-term stability of capital, offering a more flexible and comprehensive framework for modelling economic growth processes. The superior performance of fractional models in empirical studies on GDP reinforces this theoretical finding, providing strong evidence that the models are capturing a genuine, underlying economic property rather than just providing a mathematical convenience.

6. CONCLUSIONS

6.1. Summary of Findings

This paper has argued that fractional calculus is a necessary mathematical framework for modelling economic systems with "long memory" and "non-locality." By extending classical integer-order derivatives to non-integer orders, fractional models provide a more accurate and flexible tool that accounts for the influence of a system's entire history on its present state. The analysis of the fractional Solow-Swan model demonstrates how the inclusion of a fractional derivative significantly alters the dynamics of capital accumulation, offering new insights into long-term growth and capital trajectory.

6.2. Interpretation and Broader Implications

The superior empirical performance of fractional models in fitting real-world data provides strong evidence that economic systems possess an inherent memory property. Furthermore, the ability to interpret the fractional order as an intermediate between average and marginal values of economic indicators provides a crucial economic meaning to this mathematical tool. This refutes the notion that these models are merely arbitrary "fitting tools". By offering a clear and intuitive economic interpretation, the approach establishes that the models are not simply fitting a curve but are capturing a genuine, underlying economic property. The application of these models to financial markets demonstrates that this is not limited to macroeconomics but is a universal solution for problems involving long-range dependence and anomalous diffusion. This positions fractional mathematical economics as a foundational, and not just an applied, tool for the study of economic dynamics.

6.3. LIMITATIONS AND FUTURE DIRECTIONS

While promising, the field of fractional mathematical economics faces significant challenges that must be addressed for its widespread adoption. The lack of a clear, single physical or geometrical interpretation of the fractional derivative remains a topic of debate. The computational complexity and the difficulty of finding analytical solutions for fractional differential equations can be a barrier to widespread adoption and practical use. The ongoing effort to provide a clear economic interpretation of the fractional order is crucial to counter the critique that fractional models are merely "fitting tools". Future research should focus on developing more computationally efficient numerical methods and extending the economic interpretation of the fractional order to a broader range of variables and systems. The goal is to move the field from a niche application to a new foundation for the study of economic dynamics.

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