2025 Volume: 01 Issue: 07 www.ijrpa.com ISSN 2456-9995

Review Article

International Journal Research Publication Analysis

Page: 01-07

ON SOLVING NON-HOMOGENEOUS TERNARY QUINTIC DIOPHANTINE EQUATION

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Article Received: 07 November 2025

Article Revised: 27 November 2025

Published on: 17 December 2025

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DOI: https://doi-doi.org/101555/ijrpa.8069

ABSTRACT:

The non-homogeneous ternary fifth degree Diophantine equation given by $w^2+2z^2-2wx-4zx=6x^5-3x^2$ is analyzed for its patterns of non-zero distinct integral solutions.

KEYWORDS: Ternary quintic equation ,Non- Homogeneous quintic equation , Integral solutions.

INTRODUCTION:

The Diophantine equation offers an unlimited field for research due to their variety [1-4]. In particular, one may refer [5-11] for quintic equations with two ,three and five unknowns. This communication concerns with yet another interesting equation $w^2+2z^2-2wx-4zx=6x^5-3x^2$ representing non-homogeneous quintic with three unknowns for determining its infinitely many non-zero integral points.

METHOD OF ANALYSIS:

The given non-homogeneous ternary quintic Diophantine equation is

$$w^2+2z^2-2wx-4zx=6x^5-3x^2 \quad (1)$$

On completing the squares,(1) is written as

$$P^2+2Q^2=6x^5 \quad (2)$$

where

$$P = w - x, Q = z - x \tag{3}$$

By scrutiny , observe that (2) is satisfied by

$$P = 2\alpha^5, Q = \alpha^5, x = \alpha^2$$

From (3) , one obtains

$$w = \alpha^2 (2\alpha^3 + 1), z = \alpha^2 (\alpha^3 + 1)$$

The above values of x, z, w satisfy (1) . However , there are other sets of integer solutions satisfying (1). We illustrate below the process of obtaining different sets of integer solutions to (1):

Set 1:

After some algebra , it is observed that (2) is satisfied by

$$P = 6^3 m (m^2 + 2n^2)^2, Q = 6^3 n (m^2 + 2n^2)^2 \tag{4}$$

and

$$x = 6(m^2 + 2n^2) \tag{5}$$

From (4) and (3) ,we have

$$w = 6 [6^2 m (m^2 + 2n^2) + 1](m^2 + 2n^2), z = 6 [6^2 n (m^2 + 2n^2) + 1](m^2 + 2n^2) \tag{6}$$

Thus,(5) and (6) represent the integer solutions to (1).

Set 2:

Assume

$$x = m^2 + 2n^2 \tag{7}$$

Write the integer 6 on the R.H.S. of (2) as

$$6 = (2 + i\sqrt{2})(2 - i\sqrt{2}) \tag{8}$$

Assuming (7) & (8) in (2) and employing the method of factorization , consider

$$P + i\sqrt{2}Q = (2 + i\sqrt{2})(m + i\sqrt{2}n)^5 \tag{9}$$

On equating the real and imaginary parts , we have

$$P = 2 f(m,n) - 2 g(m,n), Q = f(m,n) + 2g(m,n) \tag{10}$$

where

$$f(m,n) = m^5 - 20m^3n^2 + 20mn^4,$$
$$g(m,n) = 5m^4n - 20m^2n^3 + 4n^5$$

Using (10) in (3), note that

$$z = m^2 + 2n^2 + f(m,n) + 2g(m,n), w = m^2 + 2n^2 + 2f(m,n) - 2 g(m,n) \tag{11}$$

Thus,(7) and (11) represent the integer solutions to (1).

Set 3:

Write (2) as

$$P^2 + 2Q^2 = 6x^5 * 1 \tag{12}$$

Consider 1 as

$$1 = \frac{(F(r,s) + i\sqrt{2}\ G(r,s))\ (F(r,s) - i\sqrt{2}\ G(r,s))}{[H(r,s)]^2} \tag{13}$$

where

$$F(r,s) = 2r^2 - s^2, G(r,s) = 2rs, H(r,s) = 2r^2 + s^2$$

Using (7) ,(8) and (13) in (12) and employing the method of factorization, one has

$$\begin{aligned} P + i\sqrt{2}Q &= (2 + i\sqrt{2})\ [\frac{(F(r,s) + i\sqrt{2}\ G(r,s))}{H(r,s)}](m + i\sqrt{2}n)^5 \\ &= (2 + i\sqrt{2})\ [f(m,n) + i\sqrt{2}\,g(m,n)]\ [\frac{(F(r,s) + i\sqrt{2}\ G(r,s))}{H(r,s)}] \end{aligned} \tag{14}$$

Equating the real and imaginary parts in (14) and replacing m by $H(r,s)M$ and n by $H(r,s)N$,we get

$$\begin{aligned} P &= [H(r,s)]^4 [\{2f(M,N) - 2\ g(M,N)\}F(r,s) - 2\{f(M,N) + 2\ g(M,N)\}\ G(r,s)] \\ Q &= [H(r,s)]^4 [\{2f(M,N) - 2g(M,N)\ G(r,s) + \{f(M,N) + 2g(M,N)\}\ F(r,s)] \end{aligned} \tag{15}$$

Also ,from (7) ,we have

$$x = [H(r,s)]^2 (M^2 + 2N^2) \tag{16}$$

Substituting (15) and (16) in (3) ,one obtains the corresponding values of z, w satisfying (1) .

Set 4 :

The option

$$Q = kx^2 \tag{17}$$

in (2) leads to

$$P^2 = x^4 (6x - 2k^2) \tag{18}$$

which is satisfied by

$$x=(6s^2-8s+3)\,k^2 \tag{19}$$

and

$$P=(6s-4)\ \ (6s^2-8s+3)^2\,k^5 \tag{20}$$

Using (19) in (17) , we have

$$Q=(6s^2-8s+3)^2\,k^5 \tag{21}$$

Substituting (19) , (20) and (21) in (3) , it is seen that

$$\begin{aligned} w&=(6s^2-8x+3)\,k^2\,[(6s-4)\ (6s^2-8s+3)\,k^3+1] \\ z&=(6s^2-8s+3)\,k^2\,[\,(6s^2-8s+3)\,k^3+1] \end{aligned} \tag{22}$$

Thus , (19) and (22) satisfy (1).

Note 1

It is to be noted that (18) is also satisfied by

$$x=(6s^2-4s+1)\,k^2 \tag{23}$$

and

$$P=(6s-2)\ \ (6s^2-4s+1)^2\,k^5 \tag{24}$$

Using (23) in (17) , we have

$$Q=(6s^2-4s+1)^2\,k^5 \tag{25}$$

Substituting (23) , (24) and (25) in (3) , it is seen that

$$\begin{aligned} w&=(6s^2-4x+1)\,k^2\,[(6s-2)\ (6s^2-4s+1)\,k^3+1] \\ z&=(6s^2-4s+1)\,k^2\,[\,(6s^2-4s+1)\,k^3+1] \end{aligned} \tag{26}$$

Thus , (23) and (26) satisfy (1).

Set 5 :

The option

$$P=2\ x^2 \tag{27}$$

in (2) leads to

$$\begin{aligned} &2Q^2=x^4\,(6x-4) \\ &\Rightarrow Q^2=x^4\,(3x-2) \end{aligned} \tag{28}$$

which is satisfied by

$$x=(3k^2-2k+1) \tag{29}$$

and

$$Q=(3k-1)\ (3k^2-2k+1)^2 \tag{30}$$

Using (29) in (27) , we have

$$P = 2\,(3k^2 - 2k + 1)^2 \qquad (31)$$

Substituting (29) ,(30) and (31) in (3) , it is seen that

$$\begin{aligned} w &= (3k^2 - 2k + 1)(6k^2 - 4k + 3) \\ z &= (3k^2 - 2k + 1)\,[(3k-1)\,(3k^2 - 2k + 1) + 1] \end{aligned} \qquad (32)$$

Thus , (29) and (32) satisfy (1).

Note 2

It is to be noted that (28) is also satisfied by

$$x = (3k^2 - 4k + 2) \qquad (33)$$

and

$$Q = (3k-2)\,(3k^2 - 4k + 2)^2 \qquad (34)$$

Using (33) in (27) , we have

$$P = 2\,(3k^2 - 4k + 2)^2 \qquad (35)$$

Substituting (33) ,(34) and (35) in (3) , it is seen that

$$\begin{aligned} w &= (3k^2 - 4k + 2)(6k^2 - 8k + 5) \\ z &= (3k^2 - 4k + 2)\,[(3k-2)\,(3k^2 - 4k + 2) + 1] \end{aligned} \qquad (36)$$

Thus , (33) and (36) satisfy (1).

Set 6 :

The choice

$$P = kQ \qquad (37)$$

in (2) gives

$$(k^2 + 2)\,Q^2 = 6x^5$$

which is satisfied by

$$Q = 6^3\,(k^2 + 2)^2\,\alpha^{5s} \qquad (38)$$

and

$$x = 6(k^2 + 2)\,\alpha^{2s} \qquad (39)$$

From (37) ,it is seen that

$$P = 6^3 k\,(k^2 + 2)^2\,\alpha^{5s} \qquad (40)$$

Using (38) ,(39) and (40) in (3), we have

$$\begin{aligned} w &= 6^3 k\,(k^2 + 2)^2\,\alpha^{5s} + 6(k^2 + 2)\,\alpha^{2s} \\ z &= 6^3 (k^2 + 2)^2\,\alpha^{5s} + 6\,(k^2 + 2)\,\alpha^{2s} \end{aligned} \qquad (41)$$

Thus , (1) is satisfied by (39) and (41).

Set 7 :

The choice

$$Q = k\,P \tag{42}$$

in (2) gives

$$(2k^2+1)\,P^2 = 6\,x^5$$

which is satisfied by

$$P = 6^3\,(2k^2+1)^2\,\alpha^{5s} \tag{43}$$

and

$$x = 6(2k^2+1)\,\alpha^{2s} \tag{44}$$

From (42) ,it is seen that

$$Q = 6^3 k\,(2k^2+1)^2\,\alpha^{5s} \tag{45}$$

Using (43) ,(44) and (45) in (3), we have

$$\begin{aligned} w &= 6^3\,(2k^2+1)^2\,\alpha^{5s} + 6\,(2k^2+1)\,\alpha^{2s} \\ z &= 6^3 k\,(2k^2+1)^2\,\alpha^{5s} + 6\,(2k^2+1)\,\alpha^{2s} \end{aligned} \tag{46}$$

Thus , (1) is satisfied by (44) and (46).

CONCLUSION:

In this paper, we have made an attempt to obtain all integer solutions to (1). To conclude, one may search for integer solutions to other choices of homogeneous or non-homogeneous quintic equations with multiple variables.

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