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AN INTERESTING INTEGER SEQUENCE

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ABSTRACT

This paper deals with a new integer sequence generated from the recurrence relation $x_{n+3} - 5*x_{n+2} + 8*x_{n+1} - 4*x_n = 0$ subject to the initial conditions $x_0 = 2, x_1 = 1, x_2 = 3$. Some observations among the members of the sequence are illustrated.

Keywords: Integer sequence , Relations connecting special numbers

Notations:

Mersenne number M_n : $2^n - 1$

Kynea number Ky_n : $2^{2n} + 2^{n+1} - 1$

Carol number $Carl_n$: $2^{2n} - 2^{n+1} - 1$

Woodall number W_n : $n*2^n - 1$

Cullen number Cu_n : $n*2^n + 1$

Jacobsthal number J_n : $\frac{(2^n - (-1)^n)}{3}$

Jacobsthal-Lucas number $j_n : 2^n + (-1)^n$

Thabit ibn Kurrah number Th_n : $3*2^n - 1$

INTRODUCTION

Number is the essence of mathematical calculations. Numbers have varieties of patterns and have varieties of range and richness. Many numbers exhibit fascinating properties, they form sequences, they form patterns and so on. A source of interest to both amateur and professional number theorist is the study of patterns in integers and they can be studied both geometrically and algebraically. A vital problem solving skill is the recognition of patterns in integers.In this context one may refer [1-12]. This paper presents a new integer sequence generated from the recurrence relation $x_{n+3} - 5*x_{n+2} + 8*x_{n+1} - 4*x_n = 0$ subject to the initial conditions $x_0 = 2, x_1 = 1, x_2 = 3$. Some observations among the members of the sequence are illustrated.

Method of analysis

Formulation of the sequence:

Consider the sequence of integers generated from the recurrence relation

$$x_{n+3} - 5*x_{n+2} + 8*x_{n+1} - 4*x_n = 0$$

(1)

with the initial conditions

$$x_0 = 2 , x_1 = 1, x_2 = 3$$

(2)

The auxiliary equation associated with (1) is given by

$$m^3 - 5*m^2 + 8*m - 4 = 0$$

whose solutions are

$$m = 1,2,2$$

Thus , the general solution of (1) is written as

$$x_n = C + (A\,n + B)*2^n$$

(3)

Using the conditions (2) in (3) ,we have

$$C + B = 2$$
$$C + 2*A + 2*B = 1$$
$$C + 8*A + 4*B = 3$$

Solving the above system of triple equations , one obtains the values of A,B and C to be

$$A = 2, B = -5, C = 7$$

Thus, one obtains the sequence $\{x_n\}$,whose terms are given below

$$x_n = (2*n-5)*2^n + 7 \; , n = 0,1,2,...$$

Varieties of fascinating properties

Observation 1

$$x_n = 2*n*2^n - 5*2^n + 7$$
$$= 2*[(n*2^n+1)-1] - 5*[(2^n-1)+1] + 7$$
$$x_n = 2*Cu_n - 5*M_n$$

Observation 2

$$x_n = 2*n*2^n - 5*2^n + 7$$
$$= 2*[(n*2^n-1)+1] - 5*[(2^n-1)+1] + 7$$
$$x_n = 2*W_n - 5*M_n + 4$$

Observation 3

$$x_{2n+1} = 2*(2*n+1)*2^{2n+1} - 5*2^{2n+1} + 7$$
$$= 8*n*2^{2n} - 6*2^{2n} + 7$$
$$= 4*[(2*n)*2^{2n}-1] + 4 - 6*(2^{2n}-1) + 1$$
$$x_{2n+1} = 4*W_{2n} - 6*M_{2n} + 5$$

Observation 4

From **Observation 3** , we have

$$x_{2n+1} = 4*[(2*n)*2^{2n}+1] - 4 - 6*(2^{2n}-1) + 1$$
$$\Rightarrow \; x_{2n+1} = 4*Cu_{2n} - 6*M_{2n} - 3$$

Observation 5

$$x_{2n} = 4*n*2^{2n} - 5*2^{2n} + 7$$
$$= 4*n*(2^{2n}+1) - 4*n - 5*(2^{2n}-1) - 5 + 7$$
$$x_{2n} = 4*n*j_{2n} - 5*M_{2n} - 4*n + 2$$

Observation 6

$$x_{2n} = 4*n*2^{2n} - 5*2^{2n} + 7$$
$$= 4*n*[(2^{2n}-1)+1] - 5*[(2^{2n}+1)-1] + 7$$
$$= 12*n*J_{2n} + 4*n - 5*j_{2n} + 5 + 7$$
$$x_{2n} = 12*n*J_{2n} - 5*j_{2n} + 4*n + 12$$

Observation 7

$$2^n * x_n + 2 * x_n = [(2*n-5)*2^{2n} + 7*2^n] + [(2*n-5)*2^{n+1} + 14]$$
$$= (2*n-5)*(2^{2n} + 2^{n+1}) + 7*(2^n + 2)$$
$$= (2*n-5)*(Ky_n + 1) + 7*[(2^n - 1) + 3]$$
$$= (2*n-5)*(Ky_n) + 7*M_n + 2*n + 16$$

Observation 8

$$2^n * x_n - 2 * x_n = [(2*n-5)*2^{2n} + 7*2^n] - [(2*n-5)*2^{n+1} + 14]$$
$$= (2*n-5)*(2^{2n} - 2^{n+1}) + 7*(2^n - 2)$$
$$= (2*n-5)*(Carl_n + 1) + 7*[(2^n - 1) - 1]$$
$$= (2*n-5)*(Carl_n) + 7*M_n + 2*n - 12$$

Observation 9

$$x_{n+1} - 2 * x_n = [(2*n-3)*2^{n+1} + 7] - 2*[(2*n-5)*2^n + 7]$$
$$= 2^{n+2} * n - 3*2^{n+1} + 7 - 2^{n+2} * n + 5*2^{n+1} - 14$$
$$= 2^{n+2} - 7$$
$$= (2^{n+2} - 1) - 6$$
$$\Rightarrow x_{n+1} - 2 * x_n + 6 = M_{n+2}$$

Observation 10

$$x_{n+2} - 4 * x_n = [(4*n-2)*2^{n+1} + 7] - 4*[(2*n-5)*2^n + 7]$$
$$= 2^{n+3} * n - 2*2^{n+1} + 7 - 2^{n+3} * n + 10*2^{n+1} - 28$$
$$= 2^{n+4} - 21$$
$$= (2^{n+4} - 1) - 20$$
$$\Rightarrow x_{n+2} - 4 * x_n + 20 = M_{n+4}$$

Observation 11

$$x_{n+2} - 4 * x_{n+1} + 4 * x_n = [(2*n-1)*2^{n+2} + 7] - 4*[(2*n-3)*2^{n+1} + 7] + 4*[(2*n-5)*2^n + 7]$$
$$= (8*n-4)*2^n + 7 - (16*n-24)*2^n - 28 + (8*n-20)*2^n + 28$$
$$= 7$$

Observation 12

It is worth to remind that ,a set of three non-zero integers a,b,c is said to be a Diophantine triple with property $D(n)$, if the product of any two members of the triple added with n is a perfect square.

Let

$$a = x_n - 7 \text{ ,} b = x_{n+2} - 7$$

Now , consider

$$a * b + 2^{2n+4} = [(2*n-5)*2^n]*[(2*n-1)*2^{n+2}] + 2^{2n+4}$$
$$= 2^{2n+2} *[(2*n-5)*(2*n-1)+4]$$
$$= 2^{2n+2} *(4*n^2 - 12*n + 9)$$
$$= 2^{2n+2} *(2*n-3)^2$$
$$= (x_{n+1} - 7)^2$$

Thus, the pair represents Diophantine Double with the property $D(2^{2n+4})$.

Let c be any non-zero integer such that

$$a*c + 2^{2n+4} = p^2$$
$$b*c + 2^{2n+4} = q^2$$

(4)

Eliminating c between the above two equations ,we have

$$b*p^2 - a*q^2 = 2^{2n+4}*(b-a)$$

(5) Inserting the transformations

$$p = X + a*T$$
$$q = X + b*T$$

(6) in (5) , we get the pellian equation

$$X^2 = a*b*T^2 + 2^{2n+4}$$

(7) whose smallest positive integer solutions are given by

$$T_0 = 1, X_0 = x_{n+1} - 7$$

(8)

Using (8) in (6) , we get

$$p_0 = x_n + x_{n+1} - 14 = (6*n - 11)*2^n$$

In view of (4) , we have from the first equation

$$a*c_0=(p_0)^2-2^{2n+4}=(p_0+2^{n+2})*(p_0-2^{n+2})$$
$$=[(6*n-7)*2^n]*[(6*n-15)*2^n]$$
$$=[(6*n-7)*2^n]*3*[(2*n-5)*2^n]$$
$$=[(6*n-7)*2^n]*3*a$$
$$\Rightarrow c_0=[(6*n-7)*2^n]*3=(18*n-21)*2^n$$
$$\Rightarrow c_0=x_n+2*x_{n+1}+x_{n+2}-28$$

Considering the second equation in (4) , we have

$$b*c_0+2^{2n+4}=(q_0)^2=(x_{n+1}-7+x_{n+2}-7)^2=(12*n-10)^2*2^{2n}$$
$$\Rightarrow b*c_0=(12*n-10)^2*2^{2n}-(2^{n+2})^2$$
$$=(12*n-10)^2*2^{2n}-(4*2^n)^2$$
$$=[(12*n-6)*2^n]*[(12*n-14)*2^n]$$
$$=b*(18*n-21)*2^n$$
$$\Rightarrow c_0=(18*n-21)*2^n$$

Thus , the triple $(x_n-7, x_{n+2}-7, x_n+2*x_{n+1}+x_{n+2}-28)$ is a Diophantine triple with property $D(2^{2n+4})$.

Observation 13

Let

$$a=x_n-7\ ,b=2*(x_{n+1}-7)$$

Now , consider

$$a*b+2^{2n+2}=[(2*n-5)*2^n]*[(2*n-3)*2^{n+2}]+2^{2n+2}$$
$$=2^{2n+2}*[(2*n-5)*(2*n-3)+1]$$
$$=2^{2n+2}*(4*n^2-16*n+16)$$
$$=2^{2n+2}*(2*n-4)^2$$
$$=[2^{n+1}*(2*n-4)]^2$$

Thus, the pair represents Diophantine Double with the property $D(2^{2n+2})$.

Let c be any non-zero integer such that

$$a*c+2^{2n+2}=p^2$$
$$b*c+2^{2n+2}=q^2$$

(9)

Eliminating c between the above two equations ,we have

$$b*p^2-a*q^2=2^{2n+2}*(b-a)$$

(10)

Inserting the transformations

$$p = X + a * T$$
$$q = X + b * T$$

(11) in (10) , we get the pellian equation

$$X^2 = a * b * T^2 + 2^{2n+2}$$

(12) whose smallest positive integer solutions are given by

$$T_0 = 1, X_0 = 2^{n+1} * (2 * n - 4)$$

(13) Using (13) in (11) , we get

$$p_0 = 2^{n+1} * (2 * n - 4) + (2 * n - 5) * 2^n = (6 * n - 13) * 2^n$$

In view of (9) , we have from the first equation

$$a * c_0 = (p_0)^2 - 2^{2n+2} = (p_0 + 2^{n+1}) * (p_0 - 2^{n+1})$$
$$= [(6 * n - 11) * 2^n] * [(6 * n - 15) * 2^n]$$
$$= [(6 * n - 11) * 2^n] * 3 * [(2 * n - 5) * 2^n]$$
$$= [(6 * n - 11) * 2^n] * 3 * a$$
$$\Rightarrow c_0 = [(6 * n - 11) * 2^n] * 3 = (18 * n - 33) * 2^n$$

Considering the second equation in (9) , we have

$$b * c_0 + 2^{2n+2} = (q_0)^2 = (2^n * (4 * n - 8) + 2^n * (8 * n - 12))^2 = (12 * n - 20)^2 * 2^{2n}$$
$$\Rightarrow b * c_0 = (12 * n - 20)^2 * 2^{2n} - (2^{n+1})^2$$
$$= (12 * n - 20)^2 * 2^{2n} - (2 * 2^n)^2$$
$$= [(12 * n - 18) * 2^n] * [(12 * n - 22) * 2^n]$$
$$= [6 * (2 * n - 3) * 2^n] * [2 * (6 * n - 11) * 2^n]$$
$$= b * (18 * n - 33) * 2^n$$
$$\Rightarrow c_0 = (18 * n - 33) * 2^n$$

Thus , the triple $(x_n - 7, 2 * (x_{n+1} - 7, (18 * n - 33) * 2^n)$ is a Diophantine triple with property $D(2^{2n+2})$.

Observation 14

$$\sum_{k=0}^{n-1} x_k = \sum_{k=0}^{n-1} [(2 * k - 5) * 2^k + 7]$$
$$= 2 * \sum_{k=0}^{n-1} (k * 2^k) - 5 * \sum_{k=0}^{n-1} 2^k + \sum_{k=0}^{n-1} 7$$
$$= 2 * [(n - 2) * 2^n + 2] - 5 * (2^n - 1) + 7 * n$$
$$= (2 * n - 9) * 2^n + 7 * n + 9$$

Observation 15

$$x_n = (2*n-5)*2^n + 7$$
$$= 2*[(n*2^n + 1) - 1] - 2*2^n - 3*2^n + 7$$
$$= 2*Cu_n - 2 - 2^{n+1} - [(3*2^n - 1) + 1] + 7$$
$$= 2*Cu_n - 2 - [(2^{n+1} - 1) + 1] - Th_n + 6$$
$$= 2*Cu_n - 2 - M_{n+1} - 1 - Th_n + 6$$
$$= 2*Cu_n - M_{n+1} - Th_n + 3$$

CONCLUSION

In this paper , we have presented some fascinating identities relating special integer sequence.. The readers interested in studying patterns in numbers may be motivated to obtain other forms of number patterns .

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