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# TOPOLOGICAL CONSTRAINTS ON IRREDUCIBLE DECOMPOSITIONS OF PRIME SPECTRA

***Vishnu Subhash Dhondge**

Department of Mathematics, Shri Shivaji Science and Arts College, Chikhli, District Buldhana, Affiliated to Sant Gadge Baba Amravati University, Amravati, Maharashtra, India.

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***Corresponding Author: Vishnu Subhash Dhondge**
Department of Mathematics, Shri Shivaji Science and Arts College, Chikhli, District Buldhana, Affiliated to Sant Gadge Baba Amravati University, Amravati, Maharashtra, India.

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## ABSTRACT

The prime spectrum of a commutative ring carries a natural topology in which closed sets are determined by ideals. The irreducible components of this space are controlled by minimal prime ideals, but finite irreducible decomposition is not automatic when the ring is not Noetherian. This article gives a self-contained account of the relationship between finite irreducible decomposition of a closed subset of a prime spectrum and finiteness of the minimal primes over its defining ideal. The central criterion is stated and proved using the correspondence between irreducible closed subsets and prime ideals. An explicit example in a polynomial ring in countably many variables exhibits infinitely many distinct minimal primes over a single ideal and hence a closed subset with infinitely many irreducible components. The discussion also distinguishes topological decomposition from primary decomposition and explains why quasi-compactness alone does not force a finite irreducible decomposition. The results are standard consequences of foundational commutative algebra; the purpose of the article is to assemble the criterion, proof, and non-Noetherian example into a coherent structural treatment, rather than to claim a new general theorem.

**KEYWORDS:** Prime Spectrum; Irreducible Component; Minimal Prime Ideal; Non-Noetherian Ring; Zariski Topology; Primary Decomposition.

## 1. INTRODUCTION

The prime spectrum of a commutative ring is a meeting point of algebra and topology. Its points are prime ideals, and the Zariski closed subsets are sets of prime ideals containing a

fixed ideal. This construction translates ideal-theoretic information into geometric language and makes questions about prime ideals accessible through the structure of closed subsets. Standard treatments establish that the irreducible closed subsets of a prime spectrum are precisely those of the form V(p), where p is prime, and that the irreducible components of the whole spectrum correspond to minimal prime ideals (Atiyah & Macdonald, 1969; Bourbaki, 1989; Eisenbud, 1995; Matsumura, 1986; Stacks Project, Section 10.26).

In a Noetherian setting, closed subsets admit finite decompositions into irreducible closed subsets. Without a Noetherian hypothesis, this finiteness can fail. The issue is not the existence of irreducible components: minimal prime ideals exist under the usual nonzero-ring assumptions, and irreducible components can be described by them. The issue is whether there are only finitely many such components. This distinction matters because the topological language of irreducible decomposition and the algebraic language of primary decomposition are related but not identical.

The purpose of this paper is to present a precise criterion for finite irreducible decomposition of a closed subset V(I), explain its relation to minimal prime ideals, and give a concrete non-Noetherian example where finite decomposition fails. The central equivalence is a standard result, not claimed here as a new theorem. The value of the exposition lies in the unified proof and the detailed example, which makes the obstruction to finiteness explicit. The article is organized as follows. Section 2 recalls the necessary definitions. Section 3 proves the finite-decomposition criterion. Section 4 studies a countably generated polynomial ring. Section 5 separates topological decomposition from primary decomposition, and Section 6 records consequences and limitations. The broader topological setting is that affine spectra are examples of spectral spaces, a perspective developed systematically by Hochster (1969) and Dickmann, Schwartz, and Tressl (2019).

## 2. Preliminaries

### 2.1. Prime spectrum and closed subsets

Throughout, R denotes a commutative ring with identity and I denotes an ideal of R. The prime spectrum Spec(R) is the set of all prime ideals of R. For an ideal I, define

V(I) = { p in Spec(R) : I is contained in p }.

The sets V(I) are exactly the closed subsets of Spec(R) in the Zariski topology. For ideals I and J, the standard identities include V(I) = V(rad(I)), V(IJ) = V(I) union V(J), and V(I intersect J) = V(I) union V(J). In particular, the topology depends only on the radical of an

ideal. These facts are basic properties of the spectrum and are treated in standard references such as Atiyah and Macdonald (1969) and the Stacks Project (Section 10.17).

### 2.2. Irreducibility and generic points

A nonempty topological space X is irreducible if it cannot be expressed as the union of two proper closed subsets. Equivalently, any two nonempty open subsets of X intersect. A nonempty closed subset Y of X is an irreducible component if it is irreducible and maximal, with respect to inclusion, among irreducible closed subsets.

For a prime ideal p of R, the closure of the point p in Spec(R) is V(p). It follows that V(p) is irreducible. Conversely, every irreducible closed subset of Spec(R) has the form V(p) for a unique prime ideal p. Consequently, the irreducible components of Spec(R) are precisely V(p) for minimal prime ideals p of R (Stacks Project, Section 10.26). The same correspondence applies to a closed subset V(I): its irreducible components are V(p), where p ranges over the prime ideals minimal among those containing I.

### 2.3. Minimal primes over an ideal

A prime ideal p is minimal over I if I is contained in p and no strictly smaller prime ideal contains I. Every prime ideal containing I contains a prime ideal minimal over I. This existence statement follows from the standard minimal-prime theorem, using the usual chain argument; it does not require R to be Noetherian. A Noetherian hypothesis is instead a common sufficient condition for there to be only finitely many minimal primes over each ideal (Atiyah & Macdonald, 1969; Matsumura, 1986).

## 3. A criterion for finite irreducible decomposition

The following proposition records the precise relation between a finite topological decomposition and the number of minimal primes over the defining ideal.

**Theorem 3.1**

Let R be a commutative ring and I an ideal of R. The following conditions are equivalent:

(i) The closed subset V(I) is a finite union of irreducible closed subsets of Spec(R).

(ii) There are only finitely many prime ideals minimal over I.

(iii) There are minimal prime ideals $p_1, \ldots, p_n$ over I such that $V(I) = V(p_1) \cup \ldots \cup V(p_n)$.

Proof. Suppose first that I has finitely many minimal prime ideals $p_1, \ldots, p_n$. Let q be any point of V(I), so q is a prime ideal containing I. There is a prime ideal p minimal over I with I contained in p contained in q. Since the minimal primes over I are exactly $p_1, \ldots, p_n$, the ideal p equals $p_i$ for some i. Thus q belongs to $V(p_i)$. This proves that V(I) is contained in the union

of $V(p_i)$. The reverse inclusion follows from I being contained in every $p_i$. Hence $V(I) = V(p_1)$ union … union $V(p_n)$. Each $V(p_i)$ is irreducible, so (ii) implies (iii), and (iii) implies (i).

Conversely, suppose $V(I) = Y_1$ union … union $Y_n$, where each $Y_i$ is irreducible and closed. By the irreducible-closed-subset correspondence, there is a prime ideal $q_i$ such that $Y_i = V(q_i)$. Since $V(q_i)$ is contained in V(I), we have I contained in $q_i$: indeed, every element of I belongs to every prime ideal in $V(q_i)$, and therefore belongs to the radical of $q_i$; since $q_i$ is prime, it is radical. Now let p be any prime ideal minimal over I. Because p belongs to V(I), it belongs to $V(q_i)$ for some i, and hence $q_i$ is contained in p. We have I contained in $q_i$ contained in p. Minimality of p over I forces $q_i = p$. Every minimal prime over I is therefore one of $q_1$, …, $q_n$, proving (ii). This completes the proof.

The theorem shows that the finiteness question is exactly a question about minimal prime ideals. It does not assert that every closed subset of every prime spectrum has a finite irreducible decomposition. Rather, it identifies the algebraic condition equivalent to such a decomposition for V(I). This is the standard correspondence between irreducible components and minimal primes, expressed for an arbitrary closed subset (Atiyah & Macdonald, 1969; Stacks Project, Section 10.26).

## 4. A non-Noetherian example with infinitely many components

### 4.1. The ring and ideal

Let k be a field and consider the polynomial ring in countably many indeterminates

$R = k[x_1, x_2, x_3, \ldots]$.

This ring is not Noetherian. For example, the chain of ideals

$(x_1) \subsetneq (x_1, x_2) \subsetneq (x_1, x_2, x_3) \subsetneq \ldots$

is strictly ascending. Define

$I = (x_i x_j : i \neq j)$.

Thus I is generated by all products of two distinct variables. For each positive integer n, set

$p_n = (x_i : j \neq n)$.

The quotient $R/p_n$ is isomorphic to $k[x_n]$, which is an integral domain. Hence $p_n$ is prime. Also I is contained in $p_n$ because every product $x_i x_j$ with $i \neq j$ has at least one factor whose index is not n.

### 4.2. Minimality of the primes $p_n$

Fix n and suppose q is a prime ideal satisfying $I \subseteq q \subseteq p_n$. Since $x_n$ does not belong to $p_n$, it does not belong to q. For every j different from n, the product $x_n x_j$ belongs to I and therefore to q. As q is prime and $x_n$ is not in q, it follows that $x_j$ belongs to q. This holds for every j

different from n, so $p_n$ is contained in q. The assumed reverse containment gives $q = p_n$. Therefore $p_n$ is minimal over I.

The ideals $p_n$ are pairwise distinct. If m is different from n, then $x_n$ belongs to $p_m$ but does not belong to $p_n$. Consequently, I has infinitely many distinct minimal prime ideals.

**4.3. Identifying all minimal primes**

In fact, the ideals $p_n$ are all the minimal prime ideals over I. To see this, let q be any prime ideal containing I. If every variable $x_i$ belongs to q, then $p_n$ is contained in q for every n, so q cannot be minimal over I. Otherwise, choose n such that $x_n$ does not belong to q. For each j different from n, $x_n x_j$ belongs to I and hence to q. Primality then gives $x_j \in q$ for every j different from n, so $\mathfrak{p}_n \subseteq q$. If q is minimal over I, this containment forces $q = \mathfrak{p}_n$. Thus the minimal primes over I are exactly $\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3, \ldots$.

4.4. Topological consequence

By Theorem 3.1, the closed subset V(I) cannot be expressed as a finite union of irreducible closed subsets. Its irreducible components are exactly the infinitely many closed subsets $V(p_n)$. This example demonstrates that the finite decomposition property can fail in a non-Noetherian ring, even though each component has a simple algebraic description.

The example is intentionally elementary: the prime ideals can be written down explicitly, their primality follows from a one-variable polynomial quotient, and minimality follows directly from the defining generators of I. It therefore isolates the source of the topological failure without requiring complicated calculations.

## 5. Topological decomposition versus primary decomposition

A finite irreducible decomposition of V(I) is a statement about closed subsets of Spec(R). A primary decomposition of I is an algebraic expression of I as a finite intersection of primary ideals. These are related through radicals and the identity V(I intersect J) = V(I) union V(J), but they are not the same assertion.

Suppose, for example, that I has a finite primary decomposition

$I = Q_1 \cap Q_2 \cap \ldots \cap Q_n$,

where each $Q_i$ is primary. Taking varieties gives

$V(I) = V(Q_1) \cup V(Q_2) \cup \ldots \cup V(Q_n)$.

For a primary ideal $Q_i$, its radical $\sqrt{Q_i}$ is prime, and $V(Q_i) = V(\sqrt{Q_i})$. Hence

$V(I) = V(\sqrt{Q_1}) \cup V(\sqrt{Q_2}) \cup \ldots \cup V(\sqrt{Q_n})$.

Thus a finite primary decomposition implies a finite union of irreducible closed subsets. In the standard theory, the existence of finite primary decompositions is guaranteed for ideals in

Noetherian rings (with the usual conventions and hypotheses); it is not guaranteed for arbitrary non-Noetherian rings. Conversely, a finite irreducible decomposition of V(I) records only the relevant radical or minimal-prime information and does not, by itself, provide primary components or prove a finite primary decomposition of I. For background on primary decomposition and its hypotheses, see Atiyah and Macdonald (1969), Matsumura (1986), and Eisenbud (1995).

## 6. Topological constraints and interpretation

### 6.1. Quasi-compactness is not enough

The spectrum of any ring is quasi-compact, and a closed subset of a quasi-compact space is quasi-compact. Quasi-compactness means that every open cover has a finite subcover. It does not mean that a space is a finite union of irreducible closed subsets. The example in Section 4 gives a closed subset V(I) of Spec(R) that is quasi-compact but has infinitely many irreducible components. This distinction is important because compactness-style finiteness and irreducible-decomposition finiteness are different topological properties. For the general theory of spectral spaces and their relationship with Zariski spectra, see Hochster (1969) and Dickmann et al. (2019).

### 6.2. The role of Noetherian hypotheses

Noetherianity supplies a standard sufficient condition for finite irreducible decomposition: in a Noetherian topological space, every closed subset is a finite union of irreducible closed subsets. For a Noetherian ring, its prime spectrum is a Noetherian topological space, so every V(I) has finitely many irreducible components. The converse should be stated with care: the finiteness of minimal primes for one particular ideal is weaker than the ring being Noetherian. The criterion in Theorem 3.1 is local to the ideal I and should not be confused with a global finiteness condition on all ideals.

### 6.3. What the example establishes

The countable-variable example proves that non-Noetherian rings can have ideals with infinitely many minimal primes and closed subsets with infinitely many irreducible components. It does not imply that every non-Noetherian ring has this property, nor that every ideal in a non-Noetherian ring has infinitely many minimal primes. The appropriate conclusion is conditional: finite irreducible decomposition of V(I) occurs exactly when I has finitely many minimal primes.

## 7. CONCLUSION

The topology of a prime spectrum reflects the minimal-prime structure of its underlying ring. For a closed subset V(I), finite irreducible decomposition is equivalent to the finiteness of the prime ideals minimal over I. The countably generated polynomial ring provides a direct example in which an ideal has infinitely many minimal primes, so its closed set has infinitely many irreducible components. The discussion also clarifies that this topological property is distinct from the existence of a finite primary decomposition, and that quasi-compactness alone cannot ensure finite irreducible decomposition.

The main criterion used here is established in the standard literature on commutative algebra and the topology of spectra. A further research contribution would require extending the criterion under additional structural assumptions, developing new classes of examples, or proving a genuinely new result about the distribution or behavior of minimal primes. The example presented here is best viewed as a transparent model for such investigations.

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