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AN EXPLORATION FOR INTEGER SOLUTIONS ON THE NON-HOMOGENEOUS QUATERNARY CUBIC SURFACE

$$7(x^2-y^2)=2(z^3-w^3)$$

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ABSTRACT

The third degree polynomial equation having four parameters $7(x^2-y^2)=2(z^3-w^3)$ is analyzed for integer solutions by reducing it to solvable non-homogeneous equation through suitable transformations. Observations in connection with the solutions are exhibited.

KEYWORDS: Unlike third degree equation, Quaternary third degree equation, Solutions in integer, Transformation method.

INTRODUCTION

The study of integer solutions to polynomial Diophantine equations is as old as mathematics itself. It is a treasure house in which the search for many hidden connections is a treasure hunt. It has a rich history and is an active area of current research. While attempting to understand the theory of elliptic curves , an area of attraction to mathematicians since antiquity, the polynomial Diophantine equations of third degree [1-24] are noticed. These problems motivated us for searching integer solutions to a special class of quaternary cubic polynomial Diophantine equation $7(x^2-y^2)=2(z^3-w^3)$. Observations in connection with integer solutions are exhibited

Method of analysis

The non-homogeneous quaternary cubic equation to be solved for integer solutions is

$7(x^2 - y^2) = 2(z^3 - w^3)$ (1)

The process of obtaining patterns of integer solutions to (1) is presented below:

Process 1

The substitution

$x = u + p, y = u - p, z = v + q, w = v - q, u \neq \pm p, v \neq \pm q$ (2)

in (1) gives

$7(4up) = 2(6v^2q + 2q^3)$

$\Rightarrow u = \dfrac{q(3v^2 + q^2)}{7p}$ (3)

To analyze the nature of integer solutions, one has to go in for special values to p, q in (3) and solve for u, v. In view of (2), the corresponding integer solutions to (1) are obtained.

Illustration

Choosing

$p = q = 1$ (4) in (3), we have

$u = \dfrac{(3v^2 + 1)}{7}$ (5)

Thus, taking

$v = 7s + 3$ (6)

in (5), one has

$u = 21s^2 + 18s + 4$ (7)

Using (4), (6) & (7) in (2),one has the integer solutions to (1) to be

$$\begin{array}{l} x = 21s^2 + 18s + 5, y = 21s^2 + 18s + 3 \\ z = 7s + 4, w = 7s + 2 \end{array} \quad (8)$$

A few numerical solutions are given in Table-1 below:

Table-1-Numerical solutions

s	x	y	z	w
0	5	3	4	2
1	44	42	11	9

2	125	123	18	16
3	248	246	25	23

Observations

1. $49x - 56 = 21(z-1)^2$
2. $49x - 56 = 21(w+1)^2$
3. $49y + 42 = 21(z-1)^2$
4. $49y + 42 = 21(w+1)^2$
5. $112x - 128 = 3(z^2 - w^2)^2$
6. $112y + 96 = 3(z^2 - w^2)^2$
7. The triples $(z, w, 2(z+w)), (x, y, 2(x+y))$ are such that the product of any two members of the respective triple added with 1 is a perfect square

1. Each of the following quadruples $(w, z, 2(w+z), 4(w+1)(2w+1)(2w+3))$, $(w, z, 2(w+z), 4(z-1)(2z-3)(2z-1))$, $(x, y, 2(x+y), 4(x-1)(2x-1)(2x-3))$, $(x, y, 2(x+y), 4(y+1)(2y+3)(2y+1))$ is such that the product of any two members of the quadruple added with 1 is a perfect square
2. The triples $(z, w, 2(z+w)+3), (x, y, 2(x+y)+3)$ are such that the product of any two members of the respective triple plus their sum and added with 2 is a perfect square
3. $x(w+2) = z(y+2)$
4. $y(w+2) = z(x-2)$
5. $x(w+2) - yz$ is a perfect square when $s = 14\alpha^2 + 12\alpha + 2, 14\alpha^2 + 16\alpha + 4$

Note 1

It is to be noted that (5) is also satisfied by

$v = 7s - 3, u = 21s^2 - 18s + 4$

For this choice, the corresponding integer solutions to (1) are given by

$x = 21s^2 - 18s + 5, y = 21s^2 - 18s + 3$
$z = 7s - 2, w = 7s - 4$

Remark 1

For simplicity and brevity, the integral solutions to (1) are exhibited below for the choices of p, q given by (p, q)=(2,1) ,(1,2) ,(2,2):

Choice 1: p=2 ,q=1 (9)

From (3), we have

$$u = \frac{(3v^2+1)}{14} \quad (10)$$

Thus, taking

$$v = 14s + 3 \quad (11)$$

in (10) , one has

$$u = 42s^2 + 18s + 2 \quad (12)$$

Using (9) ,(11) & (12) in (2),one has the integer solutions to (1) to be

$$\begin{aligned} & x = 42s^2 + 18s + 4 \,, y = 42s^2 + 18s \\ & z = 14s + 4, w = 14s + 2 \end{aligned} \quad (13)$$

Note 2

It is to be noted that (10) is also satisfied by

$$v = 14s - 3 \,, u = 42s^2 - 18s + 2$$

For this choice, the corresponding integer solutions to (1) are given by

$$\begin{aligned} & x = 42s^2 - 18s + 4 \,, y = 42s^2 - 18s \\ & z = 14s - 2 \,, w = 14s - 4 \end{aligned}$$

Choice 2: p=1 ,q=2 (14)

From (3) , we have

$$u = \frac{(6v^2+8)}{7} \quad (15)$$

Thus, taking

$$v = 7s + 1 \quad (16)$$

in (15), one has

$$u = 42s^2 + 12s + 2 \quad (17)$$

Using (14) ,(16) & (17) in (2),one has the integer solutions to (1) to be

$$\begin{aligned} & x = 42s^2 + 12s + 3 \,, y = 42s^2 + 12s + 1 \\ & z = 7s + 3, w = 7s - 1 \end{aligned} \quad (18)$$

Note 3

It is to be noted that (15) is also satisfied by

$$v = 7s - 1 \,, u = 42s^2 - 12s + 2$$

For this choice, the corresponding integer solutions to (1) are given by

$x = 42s^2 - 12s + 3$, $y = 42s^2 - 12s + 1$
$z = 7s + 1$, $w = 7s - 3$

Choice 3: p=2 ,q=2 (19)

From (3) , we have

$u = \frac{(3v^2 + 4)}{7}$ (20)

Thus, taking

$v = 7s + 1$ (21)

in (20) , one has

$u = 21s^2 + 6s + 1$ (22)

Using (19),(21) & (22) in (2),one has the integer solutions to (1) to be

$x = 21s^2 + 6s + 3$, $y = 21s^2 + 6s - 1$
$z = 7s + 3, w = 7s - 1$ (23)

Note 4

It is to be noted that (20) is also satisfied by

$v = 7s - 1$, $u = 21s^2 - 6s + 1$

For this choice, the corresponding integer solutions to (1) are given by

$x = 21s^2 - 6s + 3$, $y = 21s^2 - 6s - 1$
$z = 7s + 1$, $w = 7s - 3$

Process 2

Taking

$w = -z$ (24) in (1),we have

$7(x^2 - y^2) = 4z^3$ (25)

By inspection, it is seen that (25) is satisfied by the following sets of solutions:

Set 1: $x = 7^2s^3 + 1, y = 7^2s^3 - 1, z = 7s, w = -7s$

Set 2: $x = 7^2s^2 + s, y = 7^2s^2 - s, z = 7s, w = -7s$

Set 3: $x = 7s^3 + 7, y = 7s^3 - 7, z = 7s, w = -7s$

Set 4: $x = 7s^2 + 7s, y = 7s^2 - 7s, z = 7s, w = -7s$

In addition to the above solution patterns, we have other interesting patterns that are presented below:

The substitution of the transformations

$x = 7*4^2 X, y = 7*4^2 Y, z = 4*7P$ (26) in (25) leads to the non-homogeneous ternary cubic equation

$X^2 - Y^2 = P^3$ (27)

Write (27) as the system of double equations

$X + Y = P^3$
$X - Y = 1$

which is satisfied by

$P = 2c + 1$
$X = 4c^3 + 6c^2 + 3c + 1$
$Y = 4c^3 + 6c^2 + 3c$

In view of (26) & (24) , the corresponding integer solutions to (1) are given by

$x = 112(4c^3 + 6c^2 + 3c + 1)$
$y = 112(4c^3 + 6c^2 + 3c)$
$z = 28(2c + 1)$
$w = -28(2c + 1)$

Note 5

One may consider (27) as the system of double equations

$X + Y = P^2$
$X - Y = P$

which is satisfied by

$$X = \frac{P(P+1)}{2}$$
$$Y = \frac{P(P-1)}{2}$$

In view of (26) & (24) , the corresponding integer solutions to (1) are given by

$x = 56P(P+1)$
$y = 56P(P-1)$
$z = 28P$
$w = -28P$

CONCLUSION

An attempt has been made to obtain non-zero integer solutions to the non- homogeneous quaternary cubic Diophantine equation $7(x^2 - y^2) = 2(z^3 - w^3)$ by reducing it to the solvable equation through suitable transformations . One may search for other choices of transformations to reduce the degree of given equation to a lower degree equation that is solvable.

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